Peg Solitaire on Caterpillars: Making Them Solvable

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Peg solitaire is a classic game where players aim to remove pegs from a board via a jump move. In the context of graph theory, this game introduces an interesting challenge: for a given graph *G*, how many edges must be added to make it solvable in peg solitaire? This quantity is known as the minimal solvability number, ms(*G*). We study the ms(*G*) specifically for caterpillar graphs, identifying types with known minimal solvability numbers and exploring families, such as caterpillars of length 3, that remain unsolved. For this family, we provide upper bounds on the ms(*G*) and identify cases where ms(G) = 1.

Overview

This paper is broken down into 5 sections. In section 1, we give an introduction of our topic. This is followed by section 2, which discusses the known minimal solvability number of shorter caterpillars. We discuss their solvability and overall minimal solvability number. We then follow with our main results in section 3, which focuses on giving the upper bounds on the minimal solvability number for caterpillars of length 3. We finally finish our content with section 4, which gives specific examples of caterpillars of length 3 in which we know have minimal solvability number ms(G) = 1. Section 5 discusses possible open questions for further research.

Introduction

Peg solitaire is a traditional one-player game where a player has a board with cells where all but one are filled with pegs. A move in peg solitaire is known as a jump. A jump consists of three adjacent spaces, where the first two vertices have a peg, and the last vertex is a hole. A player would use the first peg to jump over the peg it is adjacent to into the empty hole. A player's objective is, through a series of jumps, to get rid of all the pegs on the board until left with only one. This game is traditionally played on an English cross shaped board (de Wiljes and Kreh, 2022).

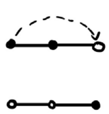

Figure 1. A jump move in peg solitaire.

Naturally, one can play on different shaped boards which can be as big or as small as desired. In fact, this combinatorial game can be played on boards that resemble arbitrary connected graphs. In 2011, Beeler and Hoilman (Beeler and Hoilman, 2011) generalized the game of peg solitaire to different graphs. Pegging moves on graphs, focusing on questions like how many pegs must one put on a graph so that, with any given distribution of pegs on any empty vertex v, one can find a sequence of jumps that places a peg at vertex v, were also studied in 2009 and 2006 respectively (Niculescu and Niculescu, 2009; Helleloid et al., 2006). A recent 2022 survey by de Wiljes and Kreh collects many papers in the peg solitaire literature. Similar to the cross board, we can play peg solitaire on a connected graph, G = (V, E), by filling in every vertex but one with a peg. One vertex must be left empty, which we will refer to as the starting hole.

To describe an arbitrary jump, let us say we are given 3 adjacent vertices, a, b, c, where the first two have pegs and the last is a hole. A jump move then uses the peg in a to jump over the second peg in b into the empty hole in c, which we will denote as $a \cdot \overrightarrow{b} \cdot c$. Upon performing this jump, the peg that was jumped over is removed, meaning we are now left with holes in a and b while c now has a peg. With every jump, a player always removes one peg and one peg only. Given a graph with n vertices for $n \in \mathbb{N}$, the maximum number of moves a player can perform is $(n - 2)$; by that point, a player would only have one peg remaining, winning the game.

Though the goal of peg solitaire is to remove all but one peg, achieving this isn't possible on every graph due to structural constraints. Some graphs inherently lack a configuration that allows a single remaining peg, making them unsolvable. To systematically study which graphs can be made solvable, we will introduce terminology and concepts developed by de Wiljes and Kreh specifically for peg solitaire on graphs (2022):

When beginning the game, we have a starting state $S \subset V$ of empty vertices, which will always be a single vertex. After playing a game, we end up with a terminal state $T \subset V$ of vertices with pegs and no more jumps are possible. We say T is associated to S if T can be reached from S through a sequence of jumps. With this in mind, we get the following:

Definition 1.1. A graph G is solvable if there is some $v \in V$ such that the starting state $S = \{v\}$ has an associated terminal state consisting of a single vertex.
Definition 1.2. A graph G is k-solvable if there is some $v \in V$ such that the starting state $S = \{v\}$ has an associated terminal state consisting of k vertices.

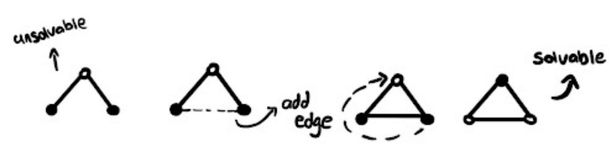

Figure 2. Adding an edge to improve solvability.

For the sake of this paper, we will only refer to a graph G as k-solvable if there exists no $j \in \mathbb{N}$ where $j < k$ such that the graph is also j-solvable, no matter where the starting hole is assigned.

In other words, the graph cannot be solved with an end state smaller than k. A k-solvable graph will have solitaire number Ps(G) = k. Knowing the solvability of graphs is important for determining the following graph invariant. The minimal solvability number ms(G) is the smallest number of edges that have to be added to a graph G in order to make it solvable (de Wiljes and Kreh, 2022) where an 'edge' represents a connection between two pegs.

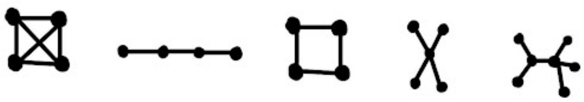

Figure 3. Examples of complete, path, cycle, star, and double star graphs.

The minimal solvability number ms(G) has been found for various graph classes such as complete graphs, path graphs, cycles, stars, and double stars (de Wiljes and Kreh, 2022). De Wiljes and Kreh point out that one "[has] to start with adding edges instead of solving the original graph first," meaning that edges must be appended first before solving the graph (2022). Furthermore, since the complete graph (a graph where all vertices are connected to each other by edges) is solvable and it is possible to add edges to any graph until it becomes complete, we know that the minimal solvability number ms(G) exists for any graph G.

De Wiljes and Kreh pose a natural question about determining the minimal solvability number ms(G) of different classes of graphs. In this paper, we focus on caterpillar graphs, a type of graph consisting of a central 'spine' of vertices with smaller branches, or 'legs,' extending from it. We examine the minimal solvability number ms(G) for various types of caterpillar graphs. Some caterpillars are isomorphic to graphs for which we know the minimal solvability number ms(G); these will be discussed and presented. More importantly, we establish precise upper bounds on the minimal solvability number ms(G) for a previously unknown caterpillar graph – specifically, caterpillars of length 3. These bounds significantly advance our understanding of the minimal solvability number ms(G) for this family. Some caterpillars of length 3 can also easily be solved with only adding one edge, so we also give examples of families of caterpillars of length 3 that have ms(G) = 1. The following proposition will be used to help with our results:

Proposition 1.1 [8, Proposition 1.1]. For every connected graph G = (V, E), ms(G) ≤ Ps(G) – 1.

Known Caterpillar Graphs

A caterpillar graph can be constructed from a path of length n by appending an arbitrary number of pendant vertices to each vertex on the path. The path with pendants attached to each of its vertices will be known as the spine of the caterpillar (Beeler et al., 2017). See **Figure 4** below:

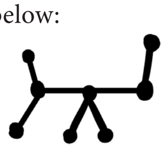

Figure 4. A caterpillar of length 3.

We want to categorize different caterpillars and determine their minimal solvability number ms(G). Through the study of the literature surrounding peg solitaire and minimal solvability number ms(G) of graphs, we are able to categorize some caterpillars and assign them a respective minimal solvability number ms(G). One simple category of caterpillars consists of caterpillars of length n with zero pendants attached to every vertex on its spine. Without the caterpillar's pendants, we are only left with the spine of the caterpillar, which by construction is a path of length n. These graphs are isomorphic to the common path graphs, in which we know their minimal solvability number.

Proposition 2.1 [3, Theorem 2.3]. Let P_n be the path on n vertices where $n \in N$. If we have P_{2n} or P_3, then this graph has ms(G) = 0. If we have P_{2n+1} for n > 1, this graph has ms(G) = 1.

This result follows immediately from the fact that paths of even length or length 3 are solvable while paths of odd length for an odd number greater than 3 are 2-solvable (Beeler and Hoilman, 2011). Combined together with Proposition 1.1, we were able to get the given result.

Following this categorization, we now consider caterpillars with pendants attached to its spine since pendants are a notable feature of a caterpillar. For the remainder of this paper, we will be referring to caterpillars of length n. When mentioning said caterpillars, the lengths refer to the length of its spine, not including any extra length that could be considered with any of the caterpillar's pendants.

That being said, let us now consider the smallest length caterpillar: a caterpillar of length one. This caterpillar will have the following notation: $Cat_1(n)$ where n is the number of pendants attached to the single vertex on its spine. These caterpillars are isomorphic to another graph with a known solvability and minimal solvability number, the star graphs. In general, a caterpillar of length 1 with n pendants is at best (n – 1)-solvable due to the nature of an empty spine vertex. If the starting hole was in the spine, no moves are possible since we would have no 2 adjacent pegs. If the starting hole was in a pendant, one can at most move from another pendant, over the spine, into the hole. Thus, (n – 1) pegs would still remain, showing it is unsolvable. Because of how unsolvable these caterpillars are, the minimal solvability number ms(G) increases with the number of pendants.

De Wiljes and Kreh determine the minimal solvability number ms(G) of star graphs by appending edges between pendants to form a sort of 'windmill blade.' Due to how they look, we will be referring to edges that connect two pendants as blades edges for the remainder of the paper. De Wiljes and Kreh ultimately prove the following result.

Proposition 2.2 [8, Corollary 2.1]. Let $Cat_1(n)$ be a caterpillar of length 1 with n pendants. If n ≥ 3, then

$$ms(Cat_1(n)) = \lceil \tfrac{n}{4} \rceil.$$

To solve caterpillar graphs of length 1 with additional blade edges, we begin by placing the initial empty hole in a pendant. From here, the solution proceeds in steps:

1. Start by jumping a peg over the central spine into the empty hole.
2. Since there exists blades, we can jump via this blade back into the empty central spine.
3. With another pendant, jump into a pendant that has a blade edge. Depending on the number of pendants in our original caterpillar of length 1, we either have another blade or our graph is solvable.

Through this process, we can achieve solvability by a sequence of systematic jumps. A detailed algorithmic jump sequence is available in (de Wiljes and Kreh, 2022).

Naturally, the next caterpillar we consider is a caterpillar of length 2. This caterpillar will have the following notation: $Cat_2(L, R)$ where L is the number of pendants attached to the left center and R is the number of pendants attached to the right center where $L, R \in \mathbb{N}$. This is the final caterpillar in which we know its minimal solvability number since it is also isomorphic to a known graph – a double star graph. The solvability of caterpillars of length 2 is known and proved by the following.

Proposition 2.3 [4, Theorem 3.1]. Let $L \geq R \geq 1$. Then, $Cat_2(L, R)$ is solvable if $L \leq R + 1$.

These caterpillars of length 2 by construction have ms(G) = 0, and the following result is known about its general minimal solvability number ms(G).

Proposition 2.4 [8, Proposition 2.5]: Let $L \geq R \geq 1$. Then,

$$ms(Cat_2(L, R)) = \left\lceil \frac{L-R-1}{4} \right\rceil.$$

A full explanation on why this works is outlined by de Wiljes and Kreh (2022). In essence, the authors consider appending edges in between left pendants and perform a specific algorithmic jump set according to whether $L - R \equiv 0, 1, 2$, or 3 mod 4. So far, we have considered a variety of different caterpillars and have given the minimal solvability number based on its construction. We considered the number of pendants and total length of the caterpillar. Up to now, the given caterpillars were isomorphic to graphs in which we knew their minimal solvability number ms(G). Next, we are going to showcase properties of the minimal solvability number ms(G) of an unknown graph.

A New Caterpillar Graph

After determining the minimal solvability number ms(G) of caterpillars of length 1 and 2, we are now interested in finding the minimal solvability number ms(G) of caterpillars of length 3. Unlike the previous caterpillars we have seen, caterpillars of length 3 are not isomorphic to a graph with a known minimal solvability number ms(G), meaning it cannot simply be presented as we have previously been doing. This raises a natural question: what is the most effective method for appending edges to caterpillars of length 3 in order to be able to solve them? Determining an optimal strategy for their placement becomes essential. To address this, we need to consider both the structure of the caterpillar and the specific arrangement of pendants, as these factors influence which configurations are solvable. We explore approaches for appending edges that minimize the number of edges needed.

In order to distinguish the vertices of a caterpillar of length 3, we are going to assign a specific name to each of the vertices, depending on where they are. Suppose you are given a caterpillar of length 3. This caterpillar will have the notation $Cat_3(L, C, R)$ where L is the number of pendants attached to the leftmost spine vertex, C is the number of pendants attached to the center spine vertex, and R is the number of pendants attached to the rightmost spine vertex. Along with the 3 spine vertices, this caterpillar has a total of L + R + C + 3 vertices. For the vertices on the spine, we will refer to the leftmost spine, center spine, and rightmost spine vertices at S_L, S_C, and S_R, respectively. The L pendant vertices will be named $l_1, l_2, \ldots, l_L$, the R pendant vertices will be named r_1 , r_2, ..., r_R, and the C pendant vertices will be named $c_1, c_2, \ldots, c_C$.

With terminology for the vertices of the caterpillar established, we now turn to determining the minimal solvability number ms(G) for caterpillars of length 3. The simplest case occurs when ms(G) = 0, indicating that the caterpillar is naturally solvable without adding edges. This leads us to a central question: under what conditions is a caterpillar of length 3 solvable?

While there is no study specifically focused on the solvability of caterpillars of length 3, we can still draw conclusions about their solvability by examining trees with diameter 4. Beeler and Walvroot (2015) demonstrate that any tree of diameter 4 can be constructed by appending pendant vertices to a star graph with n pendants. The structural relationship between caterpillars of length 3 and trees of diameter 4 allows us to leverage known results about these trees to make deductions about caterpillar solvability. In particular, the following proposition specifies the conditions under which a caterpillar of length 3 is solvable. This result was specialized for graphs with diameter at most 4.

Proposition 3.1 [5, Theorem 3.1]: Let $L \geq 2$ and $L \geq R \geq 1$. Then, $Cat_3(L, C, R)$ is solvable if

$$(L + R) - 2 \leq C \leq (L + R) + 1.$$

Furthermore, the graph is k_1–solvable, where $k_1 \leq L + R - C - 1$ if $C \leq (L + R) - 3$. Also, it is k_2–solvable, where $k_2 \leq C - L - R$ if $C \geq (L + R) + 2$.

We then can see that if $(L + R) - 2 \leq C \leq (L + R) + 1$ where $L \geq 2$ and $L \geq R \geq 1$, then the caterpillar of length 3 has ms(G) = 0. When approaching a caterpillar of length 3, one should first consider if C is within these bounds in order to see if it is originally solvable. More importantly in regard to the minimal solvability number ms(G), these caterpillars are unsolvable if $C \leq (L + R) - 3$ or if $C \geq (L + R) + 2$.

Before presenting our results, we introduce a key tool for analyzing the solvability of caterpillars of length 3. Defined by Beeler and Walvoort (2015), this tool involves configuring pegs and holes so that a specific sequence of jumps removes certain pegs while others stay constant. This setup, termed a package, is associated with an elimination process called a purge, which selectively reduces pegs on the graph. Among the various purges, the Double Star Purge is particularly valuable because it enables a systematic reduction of pendants, thereby simplifying the caterpillar's structure.

In the Double Star Purge, which applies to double star graphs (caterpillars of length 2), if the starting hole is positioned in either the left or right center, we can reduce the number of pendants on each side by $k \in \mathbb{N}$, provided $k \leq \min(L, R)$. For example, if the hole starts in the right center, we perform the Double Star Purge by first jumping from a left pendant over the left center into the hole in the right center, followed by a jump from a right pendant over the right center into the left center. After these jumps, the number of pendants on each side is reduced by 1, and the hole returns to its original position.

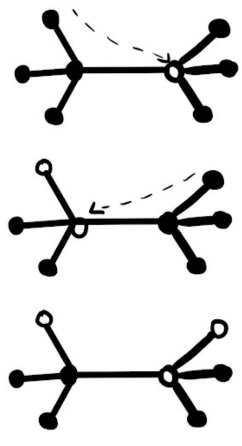

Figure 5. A Double Star Purge.

For caterpillars of length 3, we can leverage the Double Star Purge by focusing on specific sides of the caterpillar. For example, we can consider only the L left and C center pendants along with the center vertices S_L and S_C so that we work with a double star. Then, we can apply the Double Star Purge to systematically simplify this side of the caterpillar. The same principle applies if we consider the R right and C center pendants, as well as S_R and S_C. This technique allows us to progressively reduce the number of pendants and approach solvability.

Also, we will consider appending an edge connecting both S_L and S_R. This edge is one of the most efficient ones we can add since a player is able to reduce the L and R pendants by using this edge to perform Double Star Purges. Because of how it makes graphs more efficiently solvable, we will refer to the edge connecting S_L and S_R as the Double Star Edge.

We will also provide an argument as to why blade edges and the double star edge are the most efficient edges to append and are the only edges we consider in this paper. Given an unsolvable caterpillar of length 3 means that $C \leq L + R - 3$. One strategy when appending edges is to append the edges that guarantee reducing the most amount of left and right pendants with the effort of getting C outside of this bound, which would make our graph solvable. Whereas any other edge would reduce the number of pendants by 1, blade edges and the double star edge reduces the number of left and right pendants by more than one. For blades, when used correctly, it reduces the number of left or right pendants by 2 when the player jumps into S_L or S_R. Furthermore, with the double star edge, a player is able to reduce the number of pendants from both the left and right side of the caterpillar by k given that $k < R$. Doing these strategies correctly ensure that our graph becomes more solvable, showing why they may be the best strategy when appending edges and getting an accurate minimal solvability number ms(G) of caterpillars of length 3.

For the most part, caterpillars with a minimal solvability number ms(G) = 0 tend to be balanced, meaning they have a relatively equal number of left and right pendants. This pattern suggests that the closer a caterpillar is to balanced, the lower its minimal solvability number is likely to be. To better understand this relationship, we begin with the caterpillars that are most likely to have the highest minimal solvability number ms(G): those with the most extreme imbalance between left and right pendants. By establishing an upper bound in this highly unbalanced scenario, we can work closer to finding a true equality. Let us consider a caterpillar where $L = n$, $R = 1$, and $C = 0$, which is the most unbalanced that a caterpillar can get.

Theorem 3.1. Let $L \geq R \geq 1$ and $n \in \mathbb{N}$. Given $Cat_3(n, 0, 1)$, it has

$$ms(Cat_3(n, 0, 1)) \leq 1 + \left\lceil \frac{n-3}{4} \right\rceil.$$

Proof. If $n = 1$, we have P_5, which has ms(G) = 1 according to Proposition 2.1. Now, let $n \geq 2$. Put the starting hole in S_C and append the double star edge. Perform the following 2 jumps: $l_n \cdot \vec{S}_L \cdot S_C$ and $S_R \cdot \vec{S}_C \cdot S_L$, where we jump from a left pendant over the left spine into the hole in the center spine and jump from the right spine over the center spine into the left spine. The remaining pendants, peg in S_L, hole in S_R and double star edge create a sub-graph that is a caterpillar of length 2 where $L = n - 1$ and $R = 1$. By Proposition 2.4, this sub-graph has $ms(G) = \left\lceil \frac{(n-1)-1-1}{4} \right\rceil = \left\lceil \frac{n-3}{4} \right\rceil$. Since we also added the double star edge, the number of edges we added to make the caterpillar solvable was $1 + \left\lceil \frac{n-3}{4} \right\rceil$, suggesting that this caterpillar has $ms(G) \leq 1 + \left\lceil \frac{n-3}{4} \right\rceil$. ■

We, however, find that this is only an upper bound for the minimal solvability number ms(G) of this unbalanced caterpillar since we can append edges to these caterpillars a different way and get a lower ms(G). Take for instance $Cat_3(4, 0, 1)$. This caterpillar actually has ms(G) = 1 since we only append a blade edge, and it makes the entire graph solvable. We will see how in the proof below but note that there was no need to add the double star edge. In fact, through extensive case work where we played the game numerous times to the point where the number of left pendants ended up being 12, we find that we can make the upper bound tighter for these unbalanced caterpillars.

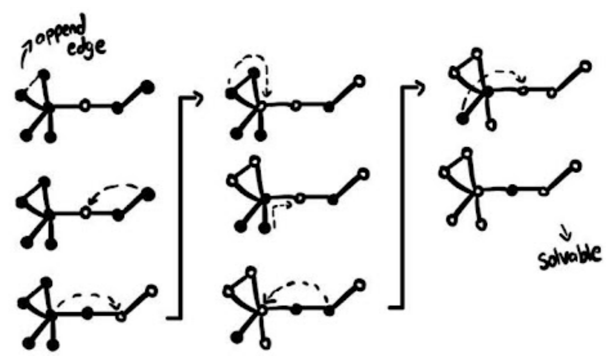

Figure 6. A game solving $Cat_3(4, 0, 1)$.

Theorem 3.2. Let $L \geq R \geq 1$ and $n \in \mathbb{N}$. Given $Cat_3(n,0,1)$, it has

$$ms(Cat_3(n, 0, 1)) \leq \left\lceil \frac{n}{4} \right\rceil.$$

Proof. If $n = 1$, we have P_5, which has ms(G) = 1 by Proposition 2.1. If $n = 2$, add an edge connecting l_1 and l_2 and put the starting hole in S_R. Perform the following 2 jumps: $S_L \cdot \vec{S}_C \cdot S_R$ and $r_1 \cdot \vec{S}_R \cdot S_C$. Using the newly appended blade edge, jump $l_1 \cdot \vec{l}_2 \cdot S_L$ and solve the graph with $r_1 \cdot \vec{S}_R \cdot S_C$. Now, let $n \geq 3$. Append $\left\lceil \frac{n}{4} \right\rceil$ edges connecting l_1 and l_2, l_3 and l_4, l_5 and l_6... so that we make separate blades. Note that we considered appending $\left\lceil \frac{n}{4} \right\rceil$ because this is the number of edges we appended to the caterpillar of length 1. We noticed that an unbalanced caterpillar of length 3 closely resembles a caterpillar of length 1. Put the starting hole in S_C and perform the following 2 jumps: $r_1 \cdot \vec{S}_R \cdot S_C$ and $S_L \cdot \vec{S}_C \cdot S_R$.

Next, using the first appended blade, perform the following jump: $l_1 \cdot \vec{l}_2 \cdot S_L$. The remaining pegs then form a caterpillar of length

2 with $n - 2$ left pendants and one right pendant where we treat S_L and S_C as the respective left and right centers. If $n < 5$, the caterpillar is solvable by Proposition 2.3.

If $n \geq 5$, more work is required. Using a non-blade pendant (like l_n), perform $l_n \cdot \vec{S}_L \cdot l_1$. This jump leaves a hole in S_L, allowing us to use our second blade and jump $l_3 \cdot \vec{l}_4 \cdot S_L$. If $n = 5$ or 6, the remaining pegs leave a solvable caterpillar of length 2, in which we solve our graph accordingly.

If $n \geq 7$, we still have an unsolvable caterpillar of length 2. In this case, however, we can use the first blade to our advantage. Jump from another non-blade pendant (like l_{n-1}) and perform $l_{n-1} \cdot \vec{S}_L \cdot l_2$. We previously added a peg in l_1, meaning out blade where l_1 and l_2 are connected is full again. Using this blade again, jump $l_1 \cdot \vec{l}_2 \cdot S_L$. If $n = 7$ or 8, the remaining pegs leave a solvable double star in which we can solve accordingly. If $n \geq 9$, we iterate the same steps we took above using our additional blades plus jumping into l_1 and l_2 to make more blades and reduce the number of left pendants. The steps we iterate are the following:

1. Using a non-blade pendant, jump into l_1.
2. Use the next blade we have not used yet to jump into S_L.
3. If still solvable, jump from a non-blade pendant over S_L into l_2.
4. Jump $l_1 \cdot \vec{l}_2 \cdot S_L$. If the graph still is not solvable, repeat from step 1. until our graph is reduced to a solvable caterpillar of length 2.

Once we do enough iterations, our graph will reduce to a solvable caterpillar of length 2. After appending $\lceil \frac{n}{4} \rceil$ edges to our caterpillar, we are able to solve our graph, suggesting that $ms(Cat_3(n, 0, 1)) \leq \lceil \frac{n}{4} \rceil$. ■

Since $\lceil \frac{n}{4} \rceil \leq 1 + \lceil \frac{n-3}{4} \rceil$, we successfully provided a tighter upper bound for this unbalanced caterpillar. Having found a tighter upper bound suggests that there may exist better strategies for adding edges, so the original algorithm provided in Theorem 3.1 is only an upper bound for unbalanced caterpillars of length 3. By this point, we have seen balanced caterpillars and very unbalanced caterpillars and gave bounds on its minimal solvability number $ms(G)$. Of course, there are also caterpillars of length 3 that are neither balanced or very unbalanced, which raises the question of whether one should append only blades or append also the double star edge to these caterpillars. Through more case work, we've decided that if we have $Cat_3(L, C, R)$ where $R > 1$, then the best method to solving these unbalanced caterpillars is by appending the double star edge and then edges based on the solvability of the caterpillar of length 2 that is sub-graphed.

In summary, the double star edge has helped us create solvable caterpillars, especially by enabling us to transition to sub-caterpillars like $Cat_2(L - 1, R)$. Then, we can use what we know to find the minimal solvability number $ms(G)$. In cases like $Cat_3(L, 0, 1)$, the process of quickly eliminating the single right pendant allowed us to simplify the problem to solving the remaining caterpillar of length 2, demonstrating the effectiveness of the blade approach in these highly asymmetric scenarios. However, as we consider caterpillars with $R > 1$, such as $Cat_3(5, 0, 2)$, our analysis suggests that relying on blades alone becomes less effective. Although we can append only a single blade edge connecting either two left or two right pendants, this configuration often leaves us with isolated pegs on one side, thus failing to solve the caterpillar.

Given these observations, it appears that once $R > 1$, the double star approach is likely more advantageous. This is supported by findings from caterpillars of length 2, where an increase in R correlated with a decrease in the minimal solvability number $ms(G)$. Thus, while further analysis may refine these conclusions (perhaps by exploring alternative edges beyond blades), the current evidence favors adopting the double star method in cases with larger values of R, as it consistently aligns with lower minimal solvability numbers.

Before proving our main result, we need to introduce the concept of hairy complete graphs. A hairy complete graph is denoted as $K_n(a_1, \ldots, a_n)$ where n is the number of vertices in the complete graph and a_i for $i = 1,\ldots,n$ is the number of pendants attached to the kth vertex of the complete graph, for $k < n$. We will now explore the solvability of hairy complete graphs. Using the notation, we get the following.

Proposition 3.2 [1, Theorem 2.1]. For a hairy complete graph $K_3(x, y, z)$ where $x \geq y \geq z$, this graph is solvable if and only if $x \leq y + z + 2$.

Beeler and Gray provide a complete discussion, but what is most important about this graph is that a caterpillar of length 3 is actually a hairy complete graph with one less edge (2016). In fact, this is why the double star edge was most helpful edge to add since caterpillars of length 3, with the addition of the double star edge, become hairy complete graphs.

We will now give an upper bound on the minimal solvability number of a general caterpillar of length 3 where $C \leq L + R - 3$. The reason we introduce the concept of $L - C - R$ is because we know that a hairy complete graph is solvable if $L \leq C + R + 2$. Thus, if $L > C + R + 2$, then it has a minimal solvability number $ms(G)$ upper bound greater than 1. We then consider if L is one more greater than this value, two more greater, and so on. So, we end up considering $L = C + R + 2 + 1$, $L = C + R + 2 + 2$, ..., $L = C + R + 2 + n$. Moving numbers around, we get the concept of $L - C - R = n$, where n is the given difference. Thus, if $n \geq 3$, then we know that the double star edge is not enough to solve our graph and that we need to append more edges. If n is anything less, we only need to append the double star edge.

Theorem 3.3. Let $L \geq R \geq 1$. Given $Cat_3(L, C, R)$ where $C \leq L + R - 3$ and $L \geq C$. If $L - C - R = n$ where $n \in \mathbb{N}$, then this caterpillar can be reduced to a case where $C = 0$ and $L' - R' = n$. This reduced caterpillar has:

$$ms(Cat_3(L', R', 0)) \leq 1 + \left\lceil \frac{L'-R'-2}{4} \right\rceil.$$

Proof. We begin this proof with an argument as why the starting hole in S_C is minimal solvability number of caterpillars of length 2. It is important to distinguish that any caterpillar with center pendants is always reducible to a caterpillar with no center pendants. We achieve this using double star purges and by having the hole in S_C. We want to be able to perform double star purges with both the left and right pendants to be able to reduce our caterpillar so that $C = 0$. Since $C \leq L + R - 3$ and $L \geq C$, we guarantee that we are going to be left with some number of right and left pendants once we reduce the number of center pendants to 0. Along with the double star edge, the remaining pegs form a caterpillar of length 2. Apart from this advantage, we can see that if the starting hole was not in S_C, we never end up with a caterpillar of length 2 with a lower minimal solvability number. Due to the nature of the minimal solvability number of caterpillars of length 2, we want to reduce the number of left pendants we have while maintaining all of our right pendants. When the hole is in S_C, we can jump $l_1 \cdot \vec{S}_L \cdot S_C$ and $S_R \cdot \vec{S}_C \cdot S_L$ to end up with $Cat_2(L-1, R)$. If the starting hole is in S_R, we at best get the same result, $Cat_2(L-1, R)$. If the starting hole is in S_L, we get a worse caterpillar in $Cat_2(L-1, R-1)$. If the hole is in one of the left or right pendants, we again only end up with at best $Cat_2(L-1, R)$ where some terminal states do not end up being double stars, which are useless in our algorithm. Therefore, you cannot get a better terminal state with a hole elsewhere and the hole in S_C allows for the initial double star purges to be done to both the center, left, and right pendants. It is thus the most efficient starting hole.

Now, suppose you have $Cat_3(L, R, C)$. Append the double star edge and have the starting hole in S_C. We know that this caterpillar is unsolvable by construction, so it has $ms(G) > 1$. If $C > 0$, we first want to reduce the caterpillar so that we end up with $C = 0$. Take the difference $L - C - R$ and call that number n. The number n lets us reduce our original caterpillar to a case where $C = 0$ and $L' - R' = n$, leaving us with $ms(Cat_3(L', R', 0))$. We can further reduce using the optimal hole in S_C and 2 jumps previously discussed to reduce our caterpillar to look like $Cat_2(L' - 1, R')$. By Proposition 2.4, this caterpillar of length 2 has $ms(Cat_2(L' - 1, R')) = \lceil \frac{L'-1-R'-1}{4} \rceil = \lceil \frac{L'-R'-2}{4} \rceil$. Therefore, with the addition of $1 + \lceil \frac{L'-R'-2}{4} \rceil$ edges, our graph becomes solvable, suggesting that $ms(Cat_3(L', R', 0)) \leq 1 + \lceil \frac{L'-R'-2}{4} \rceil$, in which $Cat_3(L', R', 0)$ was reduced from $Cat_3(L, R, C)$. ■

We will end our discussion of the minimal solvability number of general caterpillars of length 3 by providing an upper bound for the second unsolvable case. Recall from Proposition 3.1 that a caterpillar is also unsolvable if $C \geq L + R + 2$. If this is the case, the number of center pendants is always greater than the sum of the left and right pendants, meaning that double star purges would effectively rid of all the left and right pendants, leaving only center pendants. With all these double star purges, the remaining graph becomes a caterpillar of length 1, so we get the following.

Theorem 3.4. Let $L \geq R \geq 1$. Given $Cat_3(L, R, C)$, where $C \geq L + R + 2$

$$ms(Cat_3(L, R, C)) \leq \lceil \frac{C-L-R+2}{4} \rceil.$$

Proof. Start with the hole in S_L. Jump $c_1 \cdot \vec{S}_C \cdot S_L$. We are then left with $Cat_3(L, R, C-1)$ with a hole in S_C. From here, perform L left purges, getting rid of all the left pendants. Then, perform R right purges, also getting rid of all the right pendants. In this case, however, do not make the final jump from a center pendant over S_C as we want the hole to remain in S_R. Performing all those purges leaves us with a caterpillar of length 1 with $C - L - R + 2$ pendants. According to Proposition 2.2, this caterpillar has $ms(G) = \lceil \frac{C-L-R+2}{4} \rceil$. Thus, our graph becomes solvable after adding $\lceil \frac{C-L-R+2}{4} \rceil$ edges, suggesting that $ms(Cat_3(L, R, C)) \leq \lceil \frac{C-L-R+2}{4} \rceil$. ■

We have now provided a number of upper bounds for the minimal solvability number of caterpillars of length 3. However, through some case work of some unsolvable caterpillars of length 3, we find that a number of these caterpillars only need an addition of one edge to make the graph solvable. In this next section, we will give necessary parameters needed to determine if a caterpillar of length 3 has $ms(G) = 1$.

Specific Caterpillars of Length 3

We will now consider an infinite number of caterpillars that have $ms(G) = 1$, where the additional edge that was added is a blade edge or the double star edge. This will show that many unsolvable graphs become solvable with the addition of just one edge. Our main focus will be the first unsolvable case where $C \leq (L + R) - 3$. We will first consider blade edges, which was the approach we discussed for appending edges to caterpillars of length 1 and 2. Through extensive case work and referencing the k-solvability of these caterpillars, we are able to show the following:

Proposition 4.1. Let $L \geq 2$ and $L \geq R \geq 1$. A caterpillar G of length 3 such that $L + R - C = 3$ is unsolvable and has $ms(G) = 1$. Furthermore, we will show that adding an edge that connects two left pendants makes our caterpillar solvable.

Proof. Recall from Proposition 3.1 that this caterpillar is m–solvable where $m \leq (L + R - C - 1)$ since we have that $C \leq L + R - 3$, meaning it is at best 2-solvable. Thus, they have $Ps(G) = 2$, and according to Proposition 1.1, we know that this graph has $ms(G) \leq 1$. Since these graphs are unsolvable, we have that $0 < ms(G) \leq 1$, meaning it has $ms(G) = 1$. We will now work out the first 5 cases; in each of the cases, add an edge between l_1 and l_2. Later, we will show that an arbitrary caterpillar satisfying the proposition can be reduced to one of these 5 cases:

1. $L = 2, R = 1, C = 0$
   Start with a hole in S_R and complete the following jumps: $S_L \cdot \vec{S}_C \cdot S_R$, $r_1 \cdot \vec{S}_R \cdot S_C$, $l_1 \cdot \vec{l}_2 \cdot S_L$, $S_L \cdot \vec{S}_C \cdot S_R$. One peg remains, so our graph is solvable.
2. $L = 2, R = 2, C = 1$
   Start with a hole in S_R and complete the following jumps: $c_1 \cdot \vec{S}_C \cdot S_R$, $r_2 \cdot \vec{S}_R \cdot S_C$, $S_L \cdot \vec{S}_C \cdot S_R$, $l_1 \cdot \vec{l}_2 \cdot S_L$, $r_1 \cdot \vec{S}_R \cdot S_C$, $S_L \cdot \vec{S}_C \cdot S_R$. One peg remains, so our graph is solvable.
3. $L = 3, R = 1, C = 1$
   Start with a hole in S_C and complete the following jumps: $l_3 \cdot \vec{S}_L \cdot S_C$, $c_1 \cdot \vec{S}_C \cdot S_L$, $r_1 \cdot \vec{S}_R \cdot S_C$, $S_C \cdot \vec{S}_L \cdot l_3$, $l_1 \cdot \vec{l}_2 \cdot S_L$, $l_3 \cdot \vec{S}_L \cdot S_C$. One peg remains, so our graph is solvable.
4. $L = 3, R = 2, C = 2$
   Start with a hole in S_C and complete the following jumps: $r_1 \cdot \vec{S}_R \cdot S_C$, $c_1 \cdot \vec{S}_C \cdot S_R$, $r_2 \cdot \vec{S}_R \cdot S_C$, $c_2 \cdot \vec{S}_C \cdot S_R$, $l_2 \cdot \vec{S}_L \cdot S_C$, $S_R \cdot \vec{S}_C \cdot S_L$, $l_3 \cdot \vec{S}_L \cdot l_2$, $l_1 \cdot \vec{l}_2 \cdot S_L$. One peg remains, so our graph is solvable.
5. $L = 3, R = 3, C = 3$
   Start with a hole in S_C and complete the following jumps: $r_1 \cdot \vec{S}_R \cdot S_C$, $c_1 \cdot \vec{S}_C \cdot S_R$, $r_2 \cdot \vec{S}_R \cdot S_C$, $c_2 \cdot \vec{S}_C \cdot S_R$, $r_3 \cdot \vec{S}_R \cdot S_C$, $c_3 \cdot \vec{S}_C \cdot S_R$, $l_2 \cdot \vec{S}_L \cdot S_C$, $S_R \cdot \vec{S}_C \cdot S_L$, $l_3 \cdot \vec{S}_L \cdot l_2$, $l_1 \cdot \vec{l}_2 \cdot S_L$. One peg remains, so our graph is solvable.

Now, let $n, m \in N$ such that $n \geq 3$ and $m \geq 1$. Consider an arbitrary caterpillar of length 3 such that $L = n$, $R = m$, and $C = n + m - 3$. Append an edge connected l_1 and l_2 and put the starting hole in S_C. Then, do double star purges on the left and center pendants until $(n - 3)$ purges are done. This eliminates $(n - 3)$ pendants adjacent to S_L and $(n - 3)$ pendants adjacent to S_C. Next, if $m \leq 3$, do no double star purges. If $m > 3$, do double star purges on the right and center pendants until $(m - 3)$ purges are done, effectively ridding of $(m - 3)$ pendants adjacent to S_R and S_C. In either case, we reduce to one of the base cases we showed above, meaning it is still solvable. Thus, any case where we have a caterpillar of length 3 where $(L + R) - C = 3$ is unsolvable with $ms(G) = 1$. ■

We follow a similar idea for the following proposition. We group them together since they both have one extra step at the end when reducing to smaller cases.

Proposition 4.2. Let $L \geq 2$ and $L \geq R \geq 1$. A caterpillar of length 3 such that $L + R - C = 4$ or 5 is unsolvable and has ms(G) = 1. Adding an edge that connects two left pendants makes our caterpillar solvable unless $L + R - C = 5$ and $L = 3$; this latter case requires us to add an edge connecting two right pendants in order to make the graph solvable.

Proof. $L + R - C = 4$:

Similarly to Proposition 4.1, let us work out the first 4 cases and then show that an arbitrary caterpillar reduces to one of these 4 cases; in each of the cases, add an edge between l_1 and l_2:

1. $L = 2, R = 2, C = 0$
   Start with a hole in S_C and complete the following jumps: $r_1 \cdot \vec{S}_R \cdot S_C$, $S_L \cdot \vec{S}_C \cdot S_R$, $l_1 \cdot \vec{l}_2 \cdot S_L$, $r_2 \cdot \vec{S}_R \cdot S_C$, $S_L \cdot \vec{S}_C \cdot S_R$. One peg remains, so our graph is solvable.
2. $L = 3, R = 1, C = 0$
   Start with a hole in S_C and complete the following jumps: $r_1 \cdot \vec{S}_R \cdot S_C$, $S_L \cdot \vec{S}_C \cdot S_R$, $l_1 \cdot \vec{l}_2 \cdot S_L$, $l_3 \cdot \vec{S}_L \cdot S_C$, $S_R \cdot \vec{S}_C \cdot S_L$. One peg remains, so our graph is solvable.
3. $L = 3, R = 2, C = 1$
   Start with a hole in S_C and complete the following jumps: $r_1 \cdot \vec{S}_R \cdot S_C$, $S_L \cdot \vec{S}_C \cdot S_R$, $r_2 \cdot \vec{S}_R \cdot S_C$, $c_1 \cdot \vec{S}_C \cdot S_L$, $l_3 \cdot \vec{S}_L \cdot S_C$, $l_1 \cdot \vec{l}_2 \cdot S_L$, $S_L \cdot \vec{S}_C \cdot S_R$. One peg remains, so our graph is solvable.
4. $L = 3, R = 3, C = 2$
   Start with a hole in S_C and complete the following jumps: $r_1 \cdot \vec{S}_R \cdot S_C$, $c_1 \cdot \vec{S}_C \cdot S_R$, $r_2 \cdot \vec{S}_R \cdot S_C$, $S_L \cdot \vec{S}_C \cdot S_R$, $r_3 \cdot \vec{S}_R \cdot S_C$, $c_2 \cdot \vec{S}_C \cdot S_L$, $l_3 \cdot \vec{S}_L \cdot S_C$, $l_1 \cdot \vec{l}_2 \cdot S_L$, $S_L \cdot \vec{S}_C \cdot S_R$. One peg remains, so our graph is solvable.

Now, let $n, m \in \mathbb{N}$ such that $n \geq 4$ and $m \geq 1$. Consider a caterpillar of length 3 such that $L = n$, $R = m$, and $C = n + m - 4$. Append an edge connecting l_1 and l_2 and put the starting hole in S_C. Then, do double star purges on the left and center pendants until $(n - 4)$ purges are done. This eliminates $(n - 4)$ pendants adjacent to S_L and $(n - 4)$ pendants adjacent to S_C. Next, if $m \leq 4$, do no double star purges. If $m > 4$, do double star purges on the right and center pendants until $(m - 4)$ purges are done, ridding $(m - 4)$ pendants adjacent to S_R and S_C. In either case, we reduce to one of $L = 4$, $R = k$, and $C = k$ for $k = 1,2,3,4$. If $k = 1$ do one more purge on the left, and if $k \neq 1$, do 2 more purges; one on the left and one on the right. We will always reduce to one of the base cases provided, meaning our graph is solvable. Thus, any case where we have a caterpillar of length 3 where $L + R - C = 4$ has ms(G) = 1.

$L + R - C = 5$:

Again, we will work out the first 6 cases and show that an arbitrary caterpillar will reduce to one of these 6 cases. For the first two cases when $L = 3$, add an edge between r_1 and r_2:

1. $L = 3, R = 2, C = 0$
   Start with a hole in S_C and complete the following jumps: $l_1 \cdot \vec{S}_L \cdot S_C$, $S_R \cdot \vec{S}_C \cdot S_L$, $l_2 \cdot \vec{S}_L \cdot S_C$, $r_1 \cdot \vec{r}_2 \cdot S_R$, $S_R \cdot \vec{S}_C \cdot S_L$, $l_3 \cdot \vec{S}_L \cdot S_C$. One peg remains, so our graph is solvable.
2. $L = 3, R = 3, C = 1$
   Start with a hole in S_C and complete the following jumps: $r_3 \cdot \vec{S}_R \cdot S_C$, $c_1 \cdot \vec{S}_C \cdot S_R$, $l_1 \cdot \vec{S}_L \cdot S_C$, $S_R \cdot \vec{S}_C \cdot S_L$, $l_2 \cdot \vec{S}_L \cdot S_C$, $r_1 \cdot \vec{r}_2 \cdot S_R$, $S_R \cdot \vec{S}_C \cdot S_L$, $l_3 \cdot \vec{S}_L \cdot S_C$. One peg remains, so our graph is solvable.

Now, we will consider the next 4 cases. In these, add an edge between l_1 and l_2:

1. $L = 4, R = 1, C = 0$
   Start with a hole in S_C and complete the following jumps: $l_3 \cdot \vec{S}_L \cdot S_C$, $S_R \cdot \vec{S}_C \cdot S_L$, $l_4 \cdot \vec{S}_L \cdot S_C$, $l_1 \cdot \vec{l}_2 \cdot S_L$, $S_L \cdot \vec{S}_C \cdot S_R$, $r_1 \cdot \vec{S}_R \cdot S_C$. One peg remains, so our graph is solvable.
2. $L = 4, R = 2, C = 1$
   Start with a hole in S_C and complete the following jumps: $r_1 \cdot \vec{S}_R \cdot S_C$, $c_1 \cdot \vec{S}_C \cdot S_R$, $l_3 \cdot \vec{S}_L \cdot S_C$, $S_R \cdot \vec{S}_C \cdot S_L$, $l_4 \cdot \vec{S}_L \cdot S_C$, $l_1 \cdot \vec{l}_2 \cdot S_L$, $S_L \cdot \vec{S}_C \cdot S_R$, $r_2 \cdot \vec{S}_R \cdot S_C$. One peg remains, so our graph is solvable.
3. $L = 4, R = 3, C = 2$
   Start with a hole in S_C and complete the following jumps: $l_3 \cdot \vec{S}_L \cdot S_C$, $c_1 \cdot \vec{S}_C \cdot S_L$, $l_4 \cdot \vec{S}_L \cdot S_C$, $c_2 \cdot \vec{S}_C \cdot S_L$, $r_1 \cdot \vec{S}_R \cdot S_C$, $S_L \cdot \vec{S}_C \cdot S_R$, $r_2 \cdot \vec{S}_R \cdot S_C$, $l_1 \cdot \vec{l}_2 \cdot S_L$, $S_L \cdot \vec{S}_C \cdot S_R$, $r_3 \cdot \vec{S}_R \cdot S_C$. One peg remains, so our graph is solvable.
4. $L = 4, R = 4, C = 3$
   Start with a hole in S_C and complete the following jumps: $l_3 \cdot \vec{S}_L \cdot S_C$, $c_1 \cdot \vec{S}_C \cdot S_L$, $r_1 \cdot \vec{S}_R \cdot S_C$, $c_2 \cdot \vec{S}_C \cdot S_R$, $l_4 \cdot \vec{S}_L \cdot S_C$, $c_3 \cdot \vec{S}_C \cdot S_L$, $r_2 \cdot \vec{S}_R \cdot S_C$, $S_L \cdot \vec{S}_C \cdot S_R$, $r_3 \cdot \vec{S}_R \cdot S_C$, $l_1 \cdot \vec{l}_2 \cdot S_L$, $S_L \cdot \vec{S}_C \cdot S_R$, $r_4 \cdot \vec{S}_R \cdot S_C$. One peg remains, so our graph is solvable.

Let $n, m \in \mathbb{N}$ such that $n \geq 5$ and $m \geq 1$. Consider a caterpillar of length 3 such that $L = n$, $R = m$, and $C = n + m - 5$. Append an edge connected l_1 and l_2 and put the starting hole in S_C. Then, do double star purges on the left and center pendants until $(n - 5)$ purges are done. This eliminates $(n - 5)$ pendants adjacent to S_L and S_C. Next, if $m \leq 5$, do no double star purges. If $m > 5$, do double star purges on the right and center pendants until $(m - 5)$ purges are done, ridding $(m - 5)$ pendants adjacent to S_R and S_C. In either case, we reduce to one of $L = 5$, $R = k$, and $C = k$ for $k = 1,2,3,4,5$. If $k = 1$ do one more purge on the left, and if $k \neq 1$, do 2 more purges; one on both the left and right. We will always reduce to one of the base cases provided, meaning our graph is solvable. Thus, any case where we have a caterpillar of length 3 where $L + R - C = 5$ has ms(G) = 1. ■

This fact is useful to know since we are able to reduce down caterpillars that are massive and still only append one edge in order to make it solvable. This pattern unfortunately stops working when $L + R - C = 6$. Through an analysis of appending blade edges to a caterpillar where $L = 3$, $R = 3$, and $C = 0$, we see that appending one blade is not enough, no matter where the starting hole is. Checking adding a blade on either the left or right side was simple due to the symmetry of the graph. Of course, blades are not the only edges that one can add to a caterpillar of length 3, so we must consider appending edges elsewhere before concluding it has ms(G) > 1. Testing out edges like an edge connecting l_L and S_C, l_L and r_R, or r_R and S_C showed little improvement, and it still gave the graph appear to have ms(G) > 1. However, adding the double star edge helped us out tremendously. If this edge is added, then a caterpillar of length 3 where $L = 3$, $R = 3$, and $C = 0$ has ms(G) = 1. The following steps show the sequence of moves needed to solve the caterpillar with the addition of only that edge connecting S_L and S_R: Start with a hole in S_C and complete the following jumps: $l_3 \cdot \vec{S}_L \cdot S_C$, $r_3 \cdot \vec{S}_R \cdot S_L$, $l_2 \cdot \vec{S}_L \cdot S_R$, $r_2 \cdot \vec{S}_R \cdot S_L$, $l_1 \cdot \vec{S}_L \cdot S_R$, $r_1 \cdot \vec{S}_R \cdot S_L$, $S_L \cdot \vec{S}_C \cdot S_R$. This sequence of jumps effectively rids of all the pegs, but one, showing the solvability of the graph. For most of the jumps, the player ended up using double star purges between the left and right pendants since the edge connecting S_L and S_R ended up making a double star (or caterpillar of length 2) sub-graph within this caterpillar of length 3.

In fact, we can generalize the number of pendants that are on the left and right side as long as we keep $C = 0$ for now. Thus, we can get the following.

Proposition 4.3. Let $L \geq R \geq 1$. A caterpillar of length 3 such that $L = n$, $R = n$, and $C = 0$ for $n \geq 2$ has ms(G) = 1, where the additional edge added is the double star edge.

Proof. Start hole at S_C and add the double star edge. Jump $l_n \cdot \vec{S}_L \cdot S_C$. Using the double star edge, the pendants on the left and right produce a sub-graph $Cat_2(n - 1, n)$, which is solvable by

Proposition 2.3. Solving the sub-graph using double star purges leaves a peg in S_L. Finally, jump $S_L \cdot \vec{S}_C \cdot S_R$. ■

In fact, due to Proposition 2.3 and following a similar algorithm to the previous proposition:

Corollary 4.1. Let $L \geq R \geq 1$. Consider the following caterpillars of length 3:

1. $L = n, R = n - 1, C = 0$ for $n \geq 2$
2. $L = n, R = n - 2, C = 0$ for $n \geq 3$

Both of these caterpillars have ms(G) = 1 where we only append the double star edge.

Proof. $L = n, R = n - 1, C = 0$:

Start hole at S_C and add the double star edge. Jump $l_n \cdot \vec{S}_L \cdot S_C$. Using the double star edge, the pendants on the left and right produce a sub-graph $Cat_2(n - 1, n - 1)$, which is solvable by Proposition 2.3. Solving the sub-graph using double star purges leaves a peg in S_R. Finally, jump $S_R \cdot \vec{S}_C \cdot S_L$ to solve the graph.

$L = n, R = n - 2, C = 0$:

Start hole at S_C and add the double star edge. Jump $l_n \cdot \vec{S}_L \cdot S_C$ and $S_R \cdot \vec{S}_C \cdot S_L$. Using the double star edge, the remaining pendants leave a sub-graph $Cat_2(n - 1, n - 2)$ which is solvable by Proposition 2.3. Solving the sub-graph using double star purges leaves a single peg in S_R. ■

Furthermore, using our previous discussion of the solvability of hairy complete graphs, we are able to deduce the following.

Proposition 4.4. Let $L \geq 2$ and $L \geq R \geq 1$. A caterpillar of length 3 with L, R, and C left, right, and center pendants, respectively, where $C \leq (L + R) - 3$ has ms(G) = 1 if

1. $L \geq C$ and $L \leq C + R + 2$
2. $L < C$ and $C \leq L + R + 2$

In both cases, we only append the double star edge.

Proof. Appending the double star edge to this caterpillar gives us a hairy complete graph. We thus consider 2 cases. If $L \geq C$, then $L \leq C + R + 2$ for our graph to be solvable. If $L < C$, then $C \leq L + R + 2$ for our graph to be solvable. Both follow immediately from Proposition 3.2. In either case, we start the hole in S_C and reduce to a case where $L = n$, $C = 0$, and $R = n$, $n - 1$, or $n - 2$. This is possible since in either case, $C \leq (L + R) - 3$, $L \geq 2$ and $R \geq 1$. Since we only added one edge to solve the graph, this caterpillar has ms(G) = 1. ■

In fact, we can extend the hairy complete graph argument to the second unsolvable case of caterpillars of length 3.

Proposition 4.5. Let $L \geq R \geq 1$. A caterpillar of length 3 with L, R, and C left, right, and center pendants, respectively, where $C = L + R + 2$ has ms(G) = 1. We only append the double star edge.

Proof. Since for the second unsolvable case $C \geq L + R + 2$, we only consider the case where $C \geq L$. According to Proposition 3.2, by appending the double star edge, we get a hairy complete graph that is solvable if $C \leq L + R + 2$. Thus, we have that $L + R + 2 \leq C \leq L + R + 2$, meaning it is only solvable if $C = L + R + 2$. The way we solve this is by putting the starting hole in S_L and jumping $c_1 \cdot \vec{S}_C \cdot S_L$. The resulting caterpillar is $Cat_3(L, R, L + R + 1)$. By performing double star purges getting rid of the L left pendants and R right pendants, leaves us with a sub-graph with pegs in S_L, S_R, and c_{L+R+2}. Because we have the double star edge, we can perform the following jumps to solve the graph: $S_L \cdot \vec{S}_R \cdot S_C$ and $c_{L+R+2} \cdot \vec{S}_C \cdot S_R$. ■

These results show the utility of the double star edge in graph solvability and offers new pathways for investigating solvable structures in similar graph classes. By extending the framework to encompass additional pendant arrangements, this work lays the groundwork for further exploration into caterpillars of varying lengths. Moving forward, these insights could be particularly valuable when finding the minimal solvability number of other graphs that have caterpillars of length 3 as subsets.

Possible Future Work and Broader Implications

We end with some possible open questions for further research. Though we gave many upper bounds, we are interested in knowing the actual minimal solvability number ms(G) of a general caterpillar G of length 3. Of course, once that is found, how can we use that to find the ms(G) of caterpillars of length 4, 5, 6, etc. Also, one might work in proving that a certain move set/sequence of jumps is the most efficient. How do we know that an algorithm we get is the best algorithm? Moreover, one can expand and look for the minimal solvability number ms(G) of other unknown graphs like trees of diameter 4 or asters.

In retrospect, having a tight upper bound for caterpillars of length 3 furthers the study of the ms(G). Graphs can take any size and shape, and it is important to group them in families when considering this invariant. Doing so allows us to work on further graphs that have as a subset known familiar graph. In fact, this is how we were able to find the ms(G) of caterpillar graphs of length 1 and length 2. The strategies in this game can further be applied to other versions of peg solitaire. We only considered the 'jump' move, but the ms(G) might change if we consider both the 'jump' and 'merge' move of peg solitaire. Having these results gives us a starting point for this other version of the game. Lastly, we touch on results and algorithmic approaches of playing a game efficiently, and we can further take these strategies on other combinatorial games. We hope these findings contribute to any regard.

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